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An Action of A Regular Curve on $\mathbb{R}^3$ and Matlab Applications

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Abstract We define an action set of a regular curve not passing origin using a normed projection. If $\alpha(t)$ is a regular curve not passing origin, then the curve $\beta(t) = \frac{\alpha(t)}{\|\alpha(t)\|}$ is on unit sphere. $\beta(t)$ is called normed projection of $\alpha(t)$ [3]. Every point $b(t) \in \beta(t)$ defines an orthogonal matrix using Cayley's Formula. So we define an action set $R_\alpha(t) \subset SO(3)$ of $\alpha(t)$. We study in this article some important relations $\alpha(t)$ and $R_\alpha(P)$, orbit of point $P \in R^3$. At the end we give some applications in Matlab.

1. Introduction

Indicatrix of tangential, principal normal and binormal vector field of a regular curve are studied frequently [1, 4]. Many interesting properties of a space curve α in E^3 may be investigated by means of the concept of spherical indicatrix of tangent, principal normal or binormal to α [2, 7].

Every point on unit sphere defines a unit vector. This is very important for motion geometry. If $P \in S^2$ then $\|\overrightarrow{OP}\| = 1$ and $\overrightarrow{OP}$ defines a motion which its axis is a line defined by $\overrightarrow{OP}$, with rotating angle θ. To know $P = (p_1, p_2, p_3)$ is sufficient to define axis and motion matrix with rotating angle θ. For every point of regular curve α not passing origin, we can define a point on S^2 using normed projection [3]. So we can represent α on S^2. Consequently, we can define an act set (continuously motion) on R^3 using α and its spherical indicatrix.

Firstly we recall normed projection and some properties which we use.

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2. Normed Projection of a Curve on S^2

Definition 1. The mapping, $\Pi_N : R^3 - \{0\} \to S^2$, is defined as $p \to \Pi_N(p) = q$, $\overrightarrow{OQ} = \frac{\overrightarrow{OP}}{\|\overrightarrow{OP}\|}$ and is called normed projection mapping on S^2 [3].

Let $\alpha : I \subseteq R \to R^3$ be a regular curve not passing origin.

Some properties for the normed projection can be given as follows.

Property 2. Let $\alpha(t)$ be a regular curve not passing origin on a plane E passing origin.

(a) If $\alpha(t)$ is a simple open curve, $\beta(t)$ is a big circle arc.
(b) If $\alpha(t)$ is a simple closed curve, $\beta(t)$ is a big circle.
(c) The intersection of the images of the curves at E under Π_N is not empty.

Let we show the set of the regular curves not passing origin on R^3 with $R_0(R^3)$.

$$R_0(R^3) = \left\{ \alpha \mid \alpha : I \subset R \to R^3, \frac{d\alpha}{dt} \neq 0, \alpha(t) \neq 0, \text{ for all } t \right\}.$$

Proposition 3. The relation $\sim$ defined on $R_0(R^3)$ as $\alpha \sim \gamma \Leftrightarrow \Pi_N(\alpha) = \Pi_N(\gamma)$ is an equivalence relation on $R_0(R^3)$.

Proof.

Reflection Property: For $\forall\ \alpha \in R_0(R^3)$, we have $\Pi_N(\alpha) = \Pi_N(\alpha)$ so $\alpha \sim \alpha$.
Symmetry Property: If α and γ are Π_N-related, $\Pi_N(\alpha) = \Pi_N(\gamma) \Rightarrow \gamma$ and α are Π_N-related $\Rightarrow \Pi_N(\gamma) = \Pi_N(\alpha)$.
Transition Property: If α and γ are Π_N-related and γ and ξ are Π_N-related then $\Pi_N(\alpha) = \Pi_N(\gamma)$ and $\Pi_N(\gamma) = \Pi_N(\xi)$, $\Pi_N(\alpha) = \Pi_N(\xi)$.

If $\alpha(t)$ and $\gamma(t)$ are two curves, which their normed projections are the same $\beta(t)$ spherical curve, the separate property is the difference of their tangent vectors and velocities.

Namely, let

$$\beta(t) = \Pi_N(\alpha(t)) \tag{1}$$

and

$$\beta(t) = \Pi_N(\gamma(t)). \tag{2}$$

When we derive

$$\beta(t) = \frac{\alpha(t)}{\|\alpha(t)\|}, \tag{3}$$

we obtain

$$\beta'_\alpha(t) = \frac{a^2\alpha'(t) - \langle \alpha'(t), \alpha(t) \rangle\, \alpha(t)}{a^3} \tag{4}$$

where, $\|\alpha(t)\| = a$. We can do same operation for (2) and

$$\beta'_\gamma(t) = \frac{\gamma'(t)\|\gamma(t)\|^2 - (\langle\gamma'(t),\gamma(t)\rangle)\gamma(t)}{\|\gamma(t)\|^3} \tag{5}$$

is obtained. The norms of (3) and (4) are

$$\|\beta'_\alpha(t)\| = \left\|\frac{\alpha'(t)\|\alpha(t)\|^2 - \langle\alpha'(t),\alpha(t)\rangle\alpha(t)}{\|\alpha(t)\|^3}\right\| \tag{6}$$

and

$$\|\beta'_\gamma(t)\| = \left\|\frac{\gamma'(t)\|\gamma(t)\|^2 - \langle\gamma'(t),\gamma(t)\rangle\gamma(t)}{\|\gamma(t)\|^3}\right\|. \tag{7}$$

It is not required that (4) and (5), (6) and (7) are equal for $\forall\alpha(t)$ and $\gamma(t)$. □

3. Orthogonal Representation and Action Set

The set of $n \times n$ invertible matrices $GL(n,R)$ is an algebraic group under the operation of matrix multiplication Special orthogonal matrix set. $SO(n) = \{A \mid AA^T = I \text{ and } \det A = 1\}$, is an group under the operation of matrix multiplication and is called special orthogonal group [6].

$\forall\, A \in SO(n)$ defines a rotation at R^n. When $\overrightarrow{p} = \overrightarrow{OP}$ and θ are known, we can write $R_\theta \in SO(3)$. A rotating matrix around an axis $\overrightarrow{b}$ is known with components of $\overrightarrow{b}$ [6].

Rotation matrix about an arbitrary axis is defined by $\overrightarrow{b}$ with θ rotating angle

$$R_\theta = \begin{bmatrix} b_1^2(1-\cos\theta)+\cos\theta & b_1b_2(1-\cos\theta)-b_3\sin\theta & b_1b_3(1-\cos\theta)+b_2\sin\theta \\ b_1b_2(1-\cos\theta)+b_3\sin\theta & b_2^2(1-\cos\theta)+\cos\theta & b_2b_3(1-\cos\theta)-b_1\sin\theta \\ b_1b_3(1-\cos\theta)-b_2\sin\theta & b_2b_3(1-\cos\theta)+b_1\sin\theta & b_3^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

where $\overrightarrow{b} = (b_1, b_2, b_3)$ and $\|\overrightarrow{b}\| = 1$ [5].

Let

$$\alpha : I \to R^3 \tag{8}$$

be a regular curve not passing origin.

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \tag{9}$$

$(\alpha(t) \neq 0)$. If $\beta(t)$ is the normed projection of $\alpha(t)$, then

$$\Pi_N(\alpha(t)) = \beta(t), \quad \|\overrightarrow{O\beta(t)}\| = 1 \tag{10}$$

$$\beta(t) = (b_1(t), b_2(t), b_3(t)), \quad \forall\, i, b_i(t) = \frac{\alpha_i(t)}{\|\alpha(t)\|} \tag{11}$$

For $\forall t \in I$,

$$R_\theta(\beta(t)) = \begin{bmatrix} (\frac{\alpha_1(t)}{\|\alpha(t)\|})^2(1-\cos\theta)+\cos\theta & \frac{\alpha_1(t)}{\|\alpha(t)\|}\frac{\alpha_2(t)}{\|\alpha(t)\|}(1-\cos\theta)-b_3\sin\theta & \frac{\alpha_1(t)}{\|\alpha(t)\|}\frac{\alpha_3(t)}{\|\alpha(t)\|}(1-\cos\theta)+b_2\sin\theta \\ \frac{\alpha_1(t)}{\|\alpha(t)\|}\frac{\alpha_2(t)}{\|\alpha(t)\|}(1-\cos\theta)+b_3\sin\theta & (\frac{\alpha_2(t)}{\|\alpha(t)\|})^2(1-\cos\theta)+\cos\theta & \frac{\alpha_2(t)}{\|\alpha(t)\|}\frac{\alpha_3(t)}{\|\alpha(t)\|}(1-\cos\theta)-b_1\sin\theta \\ \frac{\alpha_1(t)}{\|\alpha(t)\|}\frac{\alpha_3(t)}{\|\alpha(t)\|}(1-\cos\theta)-b_2\sin\theta & \frac{\alpha_2(t)}{\|\alpha(t)\|}\frac{\alpha_3(t)}{\|\alpha(t)\|}(1-\cos\theta)+b_1\sin\theta & (\frac{\alpha_3(t)}{\|\alpha(t)\|})^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

is a rotation matrix, where $\beta(t)$ defines rotation about an fixed axis with rotation angle θ. In other words, $R_\theta(\beta(t)) = I_3 + \sin\theta.S + (1-\cos\theta)S^2$; where,

$$S = \begin{bmatrix} 0 & -\frac{\alpha_3(t)}{\|\alpha(t)\|} & \frac{\alpha_2(t)}{\|\alpha(t)\|} \\ \frac{\alpha_3(t)}{\|\alpha(t)\|} & 0 & -\frac{\alpha_1(t)}{\|\alpha(t)\|} \\ -\frac{\alpha_2(t)}{\|\alpha(t)\|} & \frac{\alpha_1(t)}{\|\alpha(t)\|} & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} R_\alpha &: I \to SO(3), \\ t &\to R_\alpha(t) \end{aligned}$$

is the representation curve on the set of the orthogonal matrix of $\alpha(t)$ regular curve.

Definition 4. $R_\alpha(t)$ is called an action set (curve) obtained from the regular curve $\alpha(t)$.

The orbit of $X = (x, y, z) \in R^3$ under R_α is certain as

$$R_\alpha(X) = (I_3 + \sin\theta S + (1-\cos\theta)S^2)X \tag{12}$$

The point of X rotates about the rotating axis, $\overrightarrow{\beta(t)}$ with θ degree for $\forall t \in I$.

The tangent vector of the orbit curve $R_\alpha(X) \subset R^3$ is obtained as

$$\begin{aligned} R'_\alpha(X) &= (I_3 + \sin\theta S + (1-\cos\theta)S^2)'X \\ &= (\sin\theta S' + 2(1-\cos\theta)SS')X \end{aligned}$$

where,

$$S' = \begin{bmatrix} 0 & \frac{-\alpha_3\|\alpha\|+\|\alpha\|'\alpha_3}{\|\alpha\|^2} & \frac{\alpha_2\|\alpha\|-\|\alpha\|'\alpha_2}{\|\alpha\|^2} \\ \frac{\alpha_3\|\alpha\|-\|\alpha\|'\alpha_3}{\|\alpha\|^2} & 0 & \frac{-\alpha_1\|\alpha\|+\|\alpha\|'\alpha_1}{\|\alpha\|^2} \\ \frac{-\alpha_2\|\alpha\|+\|\alpha\|'\alpha_2}{\|\alpha\|^2} & \frac{\alpha_1\|\alpha\|-\|\alpha\|'\alpha_1}{\|\alpha\|^2} & 0 \end{bmatrix}.$$

Consequently, speeds of these curves may be different because the tangent vectors of $\alpha(t)$, $\beta(t)$ and $R_\alpha(X)$ are different.

Now we can give some properties of a regular curve, normed projection and their representing as follows:

Property 5. The same R_α orthogonal representation for all of the cone surface which its vertex is $O = (0,0,0)$ and receives $\alpha(t)$ as the base curve is obtained.

Property 6. All of the curves which have the same base curve of cone surface are Π_N-related.

Property 7. Let $\alpha(t)$ and $\gamma(t)$ be two curves which can be taken as the base curve for a cone surface K. If $\alpha(t) = A$, $\beta(t) = B$ and A and B are on the same generated line, then

$$\Pi_N(A) = \Pi_N(B) = C \in \beta. \tag{13}$$

4. Matlab Applications

We give some applications of normed projection and their action sets using the following Matlab programme generally.

```
clear all, close all, clc
for t = -2*pi:pi/50:2*pi;
%plot3(sin(t),cos(t),t)
 grid on
axis square
c=4
axis([-c c -c c -c c])
A=cos(t);
B=sin(t);
C=0.5*t;
N =(A^2+B^2+C^2)^(1/2);
a=A/N;
b=B/N;
c=C/N;
plot3(A,B,C,'b.')
%a b c axis component
Q=60;
R=[a*a*(1-cosd(Q))+cosd(Q) a*b*(1-cosd(Q))-c*sind(Q)
     a*c*(1-cosd(Q))+b*sind(Q);
   a*b*(1-cosd(Q))+c*sind(Q) b*b*(1-cosd(Q))+cosd(Q)
     b*c*(1-cosd(Q))-a*sind(Q);
   a*c*(1-cosd(Q))-b*sind(Q) b*c*(1-cosd(Q))+a*sind(Q)
     c*c*(1-cosd(Q))+cosd(Q)];
% tr=trace(R);
% p=acosd((tr-1)/2);
% d=cotd(p/2);
% n=(d^2+a^2+b^2+c^2)^(1/2);
% QQ=[cosd(p/2);a*sind(p/2);b*sind(p/2);c*sind(p/2)]
% Q=[2*(d/n)^2-1+2*(a/n)^2   2*(a/n)*(b/n)-2*(d/n)*(c/n)
%       2*(a/n)*(c/n)+2*(d/n)*(b/n) ;
%     2*(a/n)*(b/n)+2*(d/n)*(c/n)  2*(d/n)^2-1+2*(b/n)^2
%       2*(b/n)*(c/n)-2*(d/n)*(a/n);
%     2*(a/n)*(c/n)-2*(b/n)*(d/n)  2*(b/n)*(c/n)+2*(d/n)*(a/n)
%       2*(d/n)^2-1+2*(c/n)^2]
 F=[1;2;3];
 C=[1;3;1];
```

```
 E=[1;-2;1]
 V=R*E
 K=R*C
 M=R*F
                           hold on
                           plot3(a,b,c, 'r.')
%plot3(M(1),M(2),M(3), 'r.')
%plot3(K(1),K(2),K(3), 'r.')
plot3(V(1),V(2),V(3), 'r.')
pause(0.1)
end
 k = 5;
n = 2^k-1;
theta = pi*(-n:2:n)/n;
phi = (pi/2)*(-n:2:n)'/n;
X = cos(phi)*cos(theta);
Y = cos(phi)*sin(theta);
Z = sin(phi)*ones(size(theta));
colormap([1 1 1;1 1 1])
C = hadamard(2^k);
surf(X,Y,Z,C)
axis square
```

Example 8. The normed projection of $\alpha(t) = (\cos t, \sin t, t)$, $t \in [-2\pi, 2\pi]$ on S^2 is the curve $\beta(t)$,

$$\beta(t) = \frac{1}{\sqrt{1+t^2}}(\cos t, \sin t, t) \tag{14}$$

and their Matlab figure is in Figure 1.

Example 9. The normed projection of a straight line $\alpha(t) = (-1, 0, t)$ parallel to z-axes and an orbit of a point $P(1, 2, 1)$ is given in Figure 2.

Example 10. If $\alpha(t) = (2\cos t, 2\sin t, 1)$ and $P_1 = (1, -2, 1)$, $P_2 = (1, 3, 1)$ are chosen, their normed projection and the orbits are obtained in Matlab and shown in Figure 3.

Example 11. If $\alpha(t) = (\cos t, \sin t, \frac{1}{2}t)$ and $P_1 = (1, -2, 1)$ is chosen, its normed projection and the orbit are obtained in Figure 4.

5. Conclusion

If $\alpha(t) \subset R^3$ is a regular curve not passing origin, then we have normed projection of $\alpha(t)$ onto unit sphere S^2. Then every point $P \in \alpha(t)$ is represented on S^2 and if Q is a representing point of P, then $\|\overrightarrow{OQ}\| = 1$.

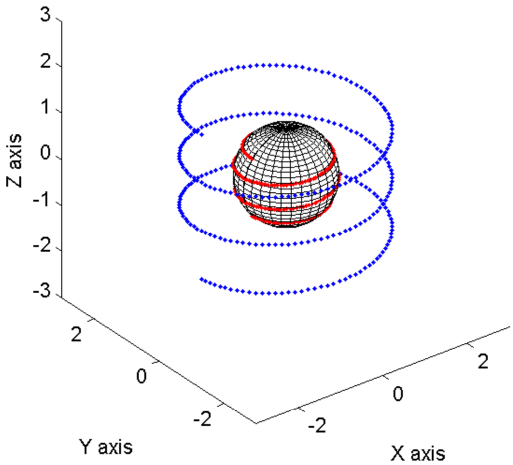

Figure 1. The normed projection of cylindrical helix

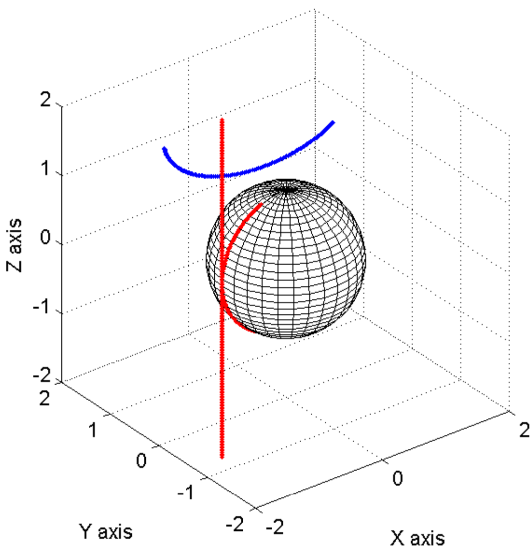

Figure 2. A projection of a line and acting

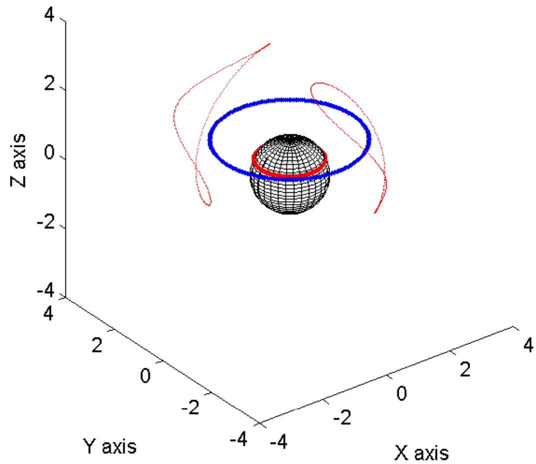

Figure 3. A projection of a circle and acting

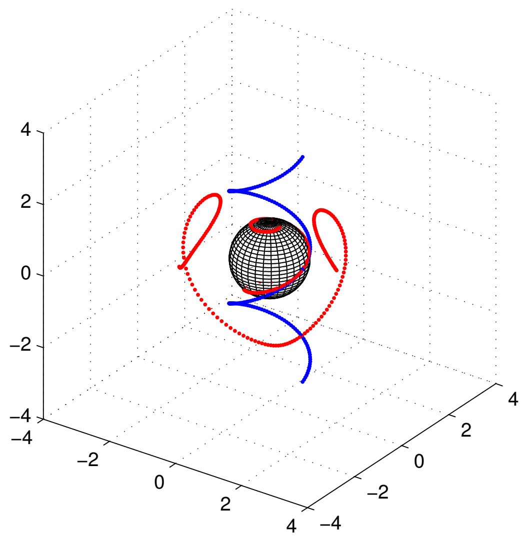

Figure 4. An acting of cylindrical helix to one point

We know that, every unit vector causes a rotating which axis is a line defined this unit vector and rotating angle θ. So, using normed projection of a regular and not passing origin curve, we can define an acting set in $SO(3)$. Thus, every regular and not passing origin curve, $\alpha(t)$, defines a motion on R^3.

In addition, if $\alpha(t)$ is on a cone surface K, every curve on K causes the same continuously rotating on R^3. The difference among the representing curve and their act sets is about velocity vectors. Choosing $\alpha(t)$ variously, we have some applications using Matlab.

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